THE BERGMAN PROJECTION AND WEIGHTED C^k ESTIMATES FOR THE CANONICAL SOLUTION TO $\bar{\partial}$ ON NON-SMOOTH DOMAINS

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ABSTRACT. We apply integral representations for functions on non-smooth strictly pseudoconvex domains, the Henkin-Leiterer domains, to derive weighted C^k estimates for the component of a given function, f, which is orthogonal to holomorphic functions in terms of C^k norms of $\bar{\partial}f$. The weights are powers of the gradient of the defining function of the domain.

1. INTRODUCTION

Let X be an n-dimensional complex manifold, equipped with a Hermitian metric, and $D \subset\subset X$ a strictly pseudoconvex domain with defining function r. We allow for singularities in the boundary, ∂D of D by permitting the possibility that dr vanishes at points on ∂D. Such domains were first studied by Henkin and Leiterer in [4], and we therefore refer to them as Henkin-Leiterer domains.

We shall make the additional assumtion that r is a Morse function. Let U be a neighborhood of ∂D. Then

$$U \cap D = \{x \in U : r(x) < 0\},$$

r with only non-degenerate critical points on U. We have

$$\partial D = \{x : r(x) = 0\},$$

and we can assume that there are finitely many critical points on bD, and none on $U \setminus bD$.

In [3], Lieb and the author studied the Bergman projection on Henkin-Leiterer domains in $\mathbb{C}^n$, and obtained weighted L^p estimates. Set $\gamma(\zeta) = |\partial r(\zeta)|$, and let us define the weighted L^p spaces by

$$L^{p,k}(D) = \left\{ f : \|f\|_{p,k}^p = \int_D \gamma^k |f|^p dV < \infty \right\}.$$

Theorem 1.1 (Ehsani-Lieb). *Let $D \subset\subset \mathbb{C}^n$ be a Henkin-Leiterer domain with a Morse defining function. The Bergman projection is continuous from $L^p(D)$ into $L^{p,k}(D)$ for $2 \leq p < \infty$ and $k = k(p)$ sufficiently large. k depends on p (with $k = 0$ for $p = 2$).*

Our purpose in this paper is to extend the results of [3]. In one direction we look to extend the results into the setting of Henkin-Leiterer domains in complex manifolds. The integral representation used in [3] relied on integral operators constructed with the use of the Henkin-Ramírez function, and is not readily adaptable

to the manifold setting. In this paper we use integral operators constructed from the Levi polynomial and the geometric distance function as in [7]. One of our main results is the reproduction of the weighted L^p estimates of [3] when D lies in a complex manifold.

We further establish C^k estimates in the theorem

Theorem 1.2. *Let $D \subset\subset X$ be a Henkin-Leiterer domain with a Morse defining function. Let $f \in L^2(D)$ and B denote the Bergman projection. Then for $\epsilon > 0$, we have*

$$\|\gamma^{(n+2)+\epsilon+k}(f - Bf)\|_{C^k} \lesssim \|\gamma^k \bar{\partial} f\|_{C^k} + \|f\|_{L^2}.$$

One of the obvious difficulties with which one comes to face in trying to establish C^k estimates on non-smooth domains is the choice of frame of vector fields with which one will work. In the case of smooth domains a special boundary chart is used in which $\omega^n = \partial r$ is part of an orthonormal frame of $(1,0)$-forms. When ∂r is allowed to vanish, the frame needs to be modified. We get around this difficulty by defining a $(1,0)$-form, ω^n by $\partial r = \gamma \omega^n$. In the dual frame of vector fields we are then faced with factors of γ in the expressions of the vector fields with respect to local coordinates, and we deal with these terms by multiplying our vector fields by a factor of γ. This ensures that when vector fields are commuted, there are no error terms which blow up at the singularity.

The author wishes to thank Ingo Lieb by whom the problem of the Bergman projection on Henkin-Leiterer domains was originally suggested. Many of ideas in the current paper were discussed with Ingo Lieb during the author's stay at the Max Plank Institute for mathematics in Bonn in 2005, and many helpful comments were taken into account.

2. Notation

With local coordinates denoted by $\zeta_1, \ldots, \zeta_n$, we equip X with a Hermitian metric

$$ds^2 = \sum_{j,k} g_{jk}(\zeta) d\zeta_j d\bar{\zeta}_k.$$

We take $\rho(x, y)$ to be a symmetric, smooth function on $X \times X$ which coincides with the geodesic distance in a neighborhood of the diagonal, Λ, and is positive outside of Λ.

We shall write $\mathcal{E}_j(x, y)$, for $j \geq 0$, for those double forms on open sets $U \subset X \times X$ such that $\mathcal{E}_j$ is smooth on U and satisfies

$$\mathcal{E}_j(x, y) \lesssim \rho^j(x, y).$$

We follow [7] to construct kernels to be used in our integral representation. Let $V^i \subset\subset U^i$ be two open coverings of the boundary ∂D. δ and ε ($\varepsilon \leq \delta$) are chosen such that

> i) V^i cover the set $\{r(x) \leq 3\delta\}$
>
> ii) $x \in V^i$ and $\rho(x, y) \leq \varepsilon$ imply $y \in U^i$
>
> iii) $|r(x)| \leq \delta$ and $\rho(x, y) \leq \varepsilon$ imply $|r(y)| < 2\delta$
>
> iv) $|r(x)| \leq 2\delta$ and $\rho(x, y) \leq \varepsilon$ imply $|r(y)| < 3\delta$.

Let (ζ^i, z^i) be local coordinates on $U^i \times U^i$. We define the Levi polynomial F^i on $U^i \times U^i$ by

$$F^i(\zeta, z) = \sum_{j=1}^n \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k),$$

where, for ease of notation, we drop the superscripts, i, on the local coordinates.

We also choose a smooth partition of unity, ξ^i such that supp $\xi^i \subset\subset V^i$ and $\sum \xi^i = 1$ on $\{|r(x)| \le 3\delta\}$. We set

$$F(x, y) = \sum_i \xi^i(x) F^i(x, y).$$

We choose a smooth symmetric patching function on $X \times X$, φ, such that $0 \le \varphi(x, y) \le 1$ and

$$\varphi(x, y) = \begin{cases} 1 & \text{for } \rho^2(x, y) \le \frac{\varepsilon}{2} \\ 0 & \text{for } \rho^2(x, y) \ge \frac{3}{4}\varepsilon, \end{cases}$$

and set

$$\phi(x, y) = \varphi(x, y)(F(x, y) - r(x)) + (1 - \varphi(x, y))\rho^2(x, y).$$

We write here the essential properties which we use in this paper. Let $D_\delta = \{x : r(x) < \delta)\}$ and $S_\delta = \{x : -\delta < r(x) < \delta)\}$.

Lemma 2.1. *There exist positive constants ϵ_0 and δ_0 such that on $\overline{S}_{\delta_0} \times \overline{D}_{\delta_0}$*

    i) $\phi = 0$ iff $x = y \in \partial D$
    ii) $|\phi| \gtrsim 1$ if $\rho(x, y) \ge \epsilon_0$
    iii) $2Re\ \phi(x, y) \gtrsim -r(x) - r(y) + \rho^2(x, y)$
    iv) $\phi - \phi^* = \mathcal{E}_3$.

Since D is strictly pseudoconvex and r is a Morse function, we can take $r_\epsilon = r + \epsilon$ for ϵ small enough, and then r_ϵ will be defining functions for smooth, strictly pseudoconvex D_ϵ.

We define $\phi_\epsilon(x, y)$ as we did $\phi(x, y)$ above with r replaced by r_ϵ. We also define

$$P(x, y) = \rho^2(x, y) + 2r(x)r(y)$$
$$P_\epsilon(x, y) = \rho^2(x, y) + 2r_\epsilon(x)r_\epsilon(y).$$

For $N \ge 0$, we let R_N denote an N-fold product, or a sum of such products, of first order vector fields applied to $r(y)$, with the notation $R_0 = 1$.

Definition 2.2. A double differential form $\mathcal{A}^\epsilon(x, y)$ on $\overline{D}_\epsilon \times \overline{D}_\epsilon$ is an *admissible* kernel, if it has the following properties:

    i) $\mathcal{A}^\epsilon$ is smooth on $\overline{D}_\epsilon \times \overline{D}_\epsilon - \Lambda_\epsilon$
    ii) For each point $(x_0, x_0) \in \Lambda_\epsilon$ there is a neighborhood $U \times U$ of (x_0, x_0) on which $\mathcal{A}^\epsilon$ or $\overline{\mathcal{A}}^\epsilon$ has the representation

(2.1) $$R_N \mathcal{E}_j P_\epsilon^{-t_0} \phi_\epsilon^{t_1} \overline{\phi}_\epsilon^{-t_2} \phi_\epsilon^{*t_3} \overline{\phi}_\epsilon^{*t_4} r_\epsilon^l r_\epsilon^{*m}$$

    with $N, j, t_0, \ldots, m$ integers and $j, t_0, l, m \ge 0$, $-t = t_1 + \cdots + t_4 \le 0$.

The above representation is of *smooth type s* for

$$s = 2n + j + \min\{2, t - l - m\} - 2(t_0 + t - l - m).$$

We define the *type* of $\mathcal{A}^\epsilon(x, y)$ to be

$$\tau = s - \max\{0, 2 - N\}.$$

$\mathcal{A}^\epsilon$ has *smooth type* $\geq s$ if at each point (x_0, x_0) there is a representation (2.1) of smooth type $\geq s$. $\mathcal{A}^\epsilon$ has *type* $\geq \tau$ if at each point (x_0, x_0) there is a representation (2.1) of type $\geq \tau$. We shall also refer to the *double type* of an operator (τ, s) if the operator is of type τ and of smooth type s.

The definition of smooth type above is taken from [7]. Here and below $(r_\epsilon(x))^* = r_\epsilon(y)$, the $*$ having a similar meaning for other functions of one variable.

For $\mathcal{A}_j^0$, we will simply write $\mathcal{A}_j$. We also denote by A_j^ϵ to be operators with kernels of the form $\mathcal{A}_j^\epsilon$. A_j will denote the operators with kernels $\mathcal{A}_j$. We use the notation $\mathcal{A}_{(j,k)}^\epsilon$ (resp. $\mathcal{A}_{(j,k)}$) to denote kernels of double type (j, k).

3. PROPERTIES OF OPERATORS

We collect in this section the various mapping properties of our operators. The proofs follow as in [1, 2].

A non-vanishing vector field, T, will be called tangential if $Tr = 0$ on $r = 0$. Near a boundary point, we choose a coordinate patch on which we have an orthogonal frame $\omega^1, \ldots, \omega^n$ of $(1, 0)$-forms with $\partial r = \gamma \omega^n$. Let $L_1, \ldots, L_n$ denote the dual frame. $L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}$, and $Y = L_n - \overline{L}_n$ are tangential vector fields. $N = L_n + \overline{L}_n$ is a normal vector field. We say a given vector field X is a smooth tangential vector field if it is a tangential field and if near each boundary point X is a combination of such vector fields $L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}, Y$, and rN with coefficients in $C^\infty(\overline{D})$.

We define the function spaces with which we will be working.

Definition 3.1. Let $0 \leq \beta$ and $0 \leq \delta$. We define
$$\|f\|_{L^{\infty,\beta,\delta}(D)} = \sup_{x \in D} |f(x)| \gamma^\beta(x) |r(x)|^\delta.$$

Definition 3.2. We set for $0 < \alpha < 1$
$$\Lambda_\alpha(D) = \{f \in L^\infty(D) \mid \|f\|_{\Lambda_\alpha} := \|f\|_{L^\infty} + \sup \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty\}.$$

We also define the spaces $\Lambda_{\alpha,\beta}$ by
$$\Lambda_{\alpha,\beta} := \{f : \|f\|_{\Lambda_{\alpha,\beta}} = \|\gamma^\beta f\|_{\Lambda_\alpha} < \infty\}.$$

Theorem 3.3. *Let T be a smooth first order tangential differential operator on D. For A an operator of type 1 we have*

$$i)\ A : L^{\infty,2+\epsilon,0}(D) \to \Lambda_{\alpha,2-\epsilon'}(D) \qquad 0 < \epsilon, \epsilon', \quad \alpha + \epsilon + \epsilon' < 1/4$$

$$ii)\ A_j : L^p(D) \to L^s(D) \quad \frac{1}{s} > \frac{1}{p} - \frac{j}{2n+2}$$

$$iii)\ \gamma^* T A : L^{\infty,2+\epsilon,0}(D) \to L^{\infty,\epsilon',\delta}(D) \qquad 1/2 < \delta < 1 \quad \epsilon < \epsilon' < 1$$

$$iv)\ A : L^{\infty,\epsilon,\delta}(D) \to L^{\infty,\epsilon',0}(D) \qquad \epsilon < \epsilon' \quad \delta < 1/2 + (\epsilon' - \epsilon)/2.$$

Theorem 3.4. *Let $1 < p < \infty$. Then*
$$A_0 : L^p(D) \to L^p(D)$$

Proof. The proof of Theorem 3.4 of [3] may be adapted to our situation to prove the inequality
$$\sup_{y \in \Omega} \int |\mathcal{A}_0(x,y)| |r(x)|^{-\delta} |r(y)|^\delta dV(x) < \infty$$

for $\delta < 1$, from which the theorem follows after an application of Hölder's inequality.

$\square$

We let $\mathcal{E}^i_{j-2n}(x,y)$, $j \geq 1$, be a kernel of the form

$$\mathcal{E}^i_{j-2n}(x,y) = \frac{\mathcal{E}_m(x,y)}{\rho^{2k}(x,y)},$$

where $m - 2k \geq j - 2n$. We denote by E_{j-2n} the corresponding isotropic operator.

Theorem 3.5. *Let T be a smooth tangential vector field. Set E to be an operator with kernel of the form $\mathcal{E}^i_{1-2n}(x,y)R_1(x)$ or $\mathcal{E}^i_{2-2n}(x,y)$. Then we have the following properties:*

> *i) $E_{j-2n} : L^p(D) \to L^s(D)$ $j \geq 1$*
>
> *ii) $E : L^{\infty,2+\epsilon,0}(D) \to \Lambda_{\alpha,2-\epsilon'}(D)$ $0 < \epsilon, \epsilon', \quad \alpha + \epsilon + \epsilon' < 1$*
>
> *iii) $\gamma^* TE : \Lambda_{\alpha,2+\epsilon}(D) \to L^{\infty,\epsilon',0}(D)$ $\epsilon < \epsilon'$*
>
> *iv) $E : L^{\infty,\epsilon,\delta}(D) \to L^{\infty,\epsilon',0}(D)$ $\epsilon < \epsilon', \quad \delta < 1/2 + (\epsilon' - \epsilon)/2$*

for any $1 \leq p \leq s \leq \infty$ with $1/s > 1/p - j/2n$.

4. Integral representation

We give here the basic integral representation. As in [3], the crucial property here is that certain operators in the integral representation, while of type 0, are of smooth type 1, hence multiplication by a factor of R_1 turns them into type 1 operators with smoothing properties.

We start with the differential forms

$$\beta(x,y) = \frac{\partial_x \rho^2(x,y)}{\rho^2(x,y)}$$

$$\alpha_\epsilon(x,y) = \xi(x)\frac{\partial r_\epsilon(x)}{\phi_\epsilon(x,y)},$$

where $\xi(x)$ is a smooth patching function which is equivalently 1 for $|r(x)| < \delta$ and 0 for $|r(x)| > \frac{3}{2}\delta$, and $\delta > 0$ is sufficiently small. We define

$$C^\epsilon = C(\alpha_\epsilon, \beta) = \sum_{\mu=0}^{n-2} \left(\frac{1}{2\pi i}\right)^n C_\mu(\alpha_\epsilon, \beta),$$

where

$$C_\mu(\alpha_\epsilon, \beta) = \alpha_\epsilon \wedge \beta \wedge (\bar{\partial}_x \alpha_\epsilon)^\mu \wedge (\bar{\partial}_x \beta)^{n-\mu-2}.$$

Denoting the Hodge $*$-operator by $*$, we then define

$$\mathcal{L}^\epsilon(x,y) = - *_x \overline{C^\epsilon(x,y)}.$$

We also write

$$K^\epsilon(x,y) = \frac{1}{(2\pi i)^n}\alpha_\epsilon \wedge (\bar{\partial}_x \alpha_\epsilon)^{n-1}$$

and

$$\Gamma^\epsilon(x,y) = \frac{(n-2)!}{2\pi^n}\frac{1}{\rho^{2n-2}}.$$

The kernels in our integral representation are defined through the following:

$$\mathcal{T}_0^\epsilon(x,y) = \vartheta_x \mathcal{L}^\epsilon(x,y) - *\overline{K}^\epsilon(x,y) + \bar{\partial}_x \Gamma^\epsilon(x,y)$$

$$\mathcal{P}_0^\epsilon(x,y) = *\partial_x \overline{K^\epsilon(x,y)}.$$

We denote the operators with kernels $\mathcal{T}_0^\epsilon$ and $\mathcal{P}_0^\epsilon$ by $\mathbf{T}_0^\epsilon$ and $\mathbf{P}_0^\epsilon$, respectively. We also define $\mathbf{T}_0$ and $\mathbf{P}_0$ to be the operators with kernels which are the limits of the kernels $\mathcal{T}_0^\epsilon$ and $\mathcal{P}_0^\epsilon$, respectively, as $\epsilon \to 0$.

Theorem 4.1. *Let $f \in L^2(D) \cap Dom\ \bar{\partial}$. Then (in the sense of distributions)*

(4.1) $$f = \mathbf{T}_0 \bar{\partial} f + \mathbf{P}_0 f + (A_{(0,2)} + E_{2-2n})\bar{\partial} f + (A_{(0,1)} + E_{1-2n})f.$$

The proof follows as in [1], up until the examination of the term (4.4) below. By carefully considering such a term, we are able to show that the resulting integral kernels in the representation are of double type $(0,1)$, whereas the analysis in [1] would only grant us kernels of double type $(-1,1)$. This information is not needed in the $q > 0$ case due to the difference in the definitions of the term $P_\epsilon(\zeta, z)$ in the cases $q = 0$ and $q > 0$.

Sketch of proof. We first prove the corresponding statement for the domain D_ϵ and in the case $f \in C^1(\overline{D})$.

Our starting point is the Bochner-Martinelli-Koppelman (BMK) formula for $f \in C^1(\overline{D}_\epsilon)$. Let B_0 be defined by

$$B_0 = \Omega_0(\beta) = \frac{1}{(2\pi i)^n} \beta \wedge (\bar{\partial}_\zeta \beta)^{n-1}.$$

Then for $y \in D_\epsilon$

(4.2) $$f(y) = \int_{\partial D_\epsilon} f(x) \wedge B_0(x,y) - \int_{D_\epsilon} \bar{\partial} f(x) \wedge B_0(x,y)$$
$$+ (f(x), \mathcal{E}_{1-2n}(x,y)) + (\bar{\partial} f(x), \mathcal{E}_{2-2n}(x,y)).$$

Define the kernels $K_0^\epsilon(x,y)$ by $K_0^\epsilon(x,y) = \Omega_0(\alpha_\epsilon)$. We then proceed to replace the boundary integral in the BMK formula by

$$\int_{\partial D_\epsilon} f \wedge K_0^\epsilon.$$

Let $x_0 \in \partial D_\epsilon$ be a fixed point and U a sufficiently small neighborhood of x_0 on which ζ is a local coordinate map. $F(\zeta, z)$ vanishes on the diagonal of $\overline{U} \times \overline{U}$, so Hefer's theorem applies to give us

$$F(\zeta, z) = \sum_{j=1}^n h_j(\zeta, z)(\zeta_j - z_j).$$

We set

$$\alpha^0(\zeta, z) = \frac{\sum_{j=1}^n h_j(\zeta, z)d\zeta_j}{F}.$$

With the metric given by

$$ds^2 = \sum g_{jk}(\zeta)dz_j d\overline{z}_k,$$

we define

$$b^0(\zeta, z) = \sum_{j,k=1}^{n} g_{jk}(\overline{\zeta}_k - \overline{z}_k)d\zeta_j$$

$$R^2(\zeta, z) = \sum_{j,k=1}^{n} g_{jk}(\zeta_j - z_j)(\overline{\zeta}_k - \overline{z}_k)$$

$$\beta^0(\zeta, z) = \frac{b^0(\zeta, z)}{R^2(\zeta, z)}.$$

With use of the transition kernels C defined above, we have via Koppelman's homotopy formula

$$\Omega_0(\beta^0) = \Omega_0(\alpha^0) - \bar{\partial}_\zeta C(\alpha^0, \beta^0).$$

On $(\partial D \cap U) \times U$ we have

$$\Omega_0(\alpha^\epsilon) = \Omega_0(\alpha_0) + \frac{\mathcal{E}_1(\zeta, z)}{\phi_\epsilon(\zeta, z)^n}$$

and on $U \times U$ we have

$$\Omega_0(\beta) = \frac{R^{2n}}{\rho^{2n}}\Omega_0(\beta^0) + \mathcal{E}_{2-2n}.$$

Thus we write

$$\Omega_0(\beta) = \frac{R^{2n}}{\rho^{2n}}\Omega_0(\beta^0) + \mathcal{E}_{2-2n}$$

$$= \frac{R^{2n}}{\rho^{2n}}(\Omega_0(\beta^0) - \Omega_0(\alpha^0)) + \Omega_0(\alpha^0) + \frac{\mathcal{E}_{2n+1}}{\rho^{2n}}\Omega_0(\alpha^0) + \mathcal{E}_{2-2n},$$

by which it then follows from the homotopy formula and the relations between b^0 and ρ^2 and R^2, exactly as it was obtained in [6], that we have

$$(4.3) \qquad \Omega_0(\beta) = \Omega_0(\alpha_\epsilon) - \bar{\partial}_\zeta C^\epsilon + (\Omega_0(\alpha^0) - \Omega_0(\alpha_\epsilon)) + \frac{\mathcal{E}_{2n+1}}{\rho^{2n}}\Omega_0(\alpha^0)$$

$$+ \mathcal{E}_{2-2n} + \bar{\partial}_\zeta\left[\left(C\left(\alpha^0, \frac{\partial\rho^2 + \mathcal{E}_2}{\rho^2}\right) - C(\alpha_\epsilon, \beta)\right)\right.$$

$$\left. + \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{3+2\mu}}{(\rho^2)^{1+\mu}}C_\mu\left(\alpha^0, \frac{\partial\rho^2 + \mathcal{E}_2}{\rho^2}\right)\right]$$

$$+ \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{4+2\mu}}{(\rho^2)^{2+\mu}}C_\mu\left(\alpha^0, \frac{\partial\rho^2 + \mathcal{E}_2}{\rho^2}\right).$$

We now work with $\zeta \in \partial D_\epsilon$ so that $F = \phi_\epsilon$.

We have

$$\frac{\mathcal{E}_{2n+1}}{\rho^{2n}}\Omega_0(\alpha^0) = R_1\frac{\mathcal{E}_{2n+1}}{\phi_\epsilon^n \rho^{2n}}$$

and

$$\Omega_0(\alpha_\epsilon) - \Omega_0(\alpha^0) = \frac{\mathcal{E}_1}{\phi_\epsilon^n}.$$

Furthermore,

$$C_\mu\left(\alpha^0, \frac{\partial\rho^2 + \mathcal{E}_2}{\rho^2}\right) = R_1\frac{\mathcal{E}_1}{\phi_\epsilon^{1+\mu}(\rho^2)^{n-1-\mu}},$$

and thus

$$\bar{\partial}_\zeta \left(\sum_\mu \frac{\mathcal{E}_{3+2\mu}}{(\rho^2)^{1+\mu}} C_\mu \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right) \right) = \bar{\partial}_\zeta \left(R_1 \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} \right)$$

$$= \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + R_1 \frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + R_1 \frac{\mathcal{E}_{5+2\mu} + \mathcal{E}_{4+2\mu} \wedge \bar{\partial}_\zeta r_\epsilon}{\phi_\epsilon^{2+\mu} \rho^{2n}}.$$

We have

(4.4) $$C_\mu \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right) - C_\mu(\alpha_\epsilon, \beta) = \frac{\mathcal{E}_2}{\phi_\epsilon^{1+\mu} (\rho^2)^{n-1-\mu}},$$

and by carefully handling this term we obtain a slight improvement, with tangible benefits, over its treatment in [6] and even in [1], where the additional information has no effect. In our relating boundary integrals to volume integrals, with the use of Stoke's Theorem, we replace ρ^2 with P_ϵ, as they are equivalent on ∂D_ϵ, in the kernels. We therefore, use here the relation

$$\bar{\partial}_\zeta \left(C_\mu \left(\alpha^0, \frac{\partial \rho^2 + \mathcal{E}_2}{\rho^2} \right) - C_\mu(\alpha_\epsilon, \beta) \right) = \bar{\partial}_\zeta \left(\frac{\mathcal{E}_2}{\phi_\epsilon^{1+\mu} P_\epsilon^{n-1-\mu}} \right) + \frac{\mathcal{E}_2 r_\epsilon^* \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{1+\mu} P_\epsilon^{n-\mu}}$$

for $\zeta \in \partial D_\epsilon$. The upshot is that the integral kernels in our final expression which stem from the term (4.4) are of double type $(0,1)$.

(4.3) can thus be written

$$\Omega_0(\beta) = \Omega_0(\alpha_\epsilon) - \bar{\partial}_\zeta C(\alpha_\epsilon, \beta) + \mathcal{E}_{2-2n}$$

$$+ \sum_{\mu=0}^{n-2} \bar{\partial}_\zeta \left(\frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} \right) + r_\epsilon^* \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{2+2\mu} \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + \frac{\mathcal{E}_1}{\phi_\epsilon^n}$$

$$+ R_1 \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + R_1 \sum_{\mu=0}^{n-2} \frac{\mathcal{E}_{5+2\mu} + \mathcal{E}_{4+2\mu} \wedge \bar{\partial}_\zeta r_\epsilon}{\phi_\epsilon^{2+\mu} \rho^{2n}},$$

again for $\zeta \in \partial D_\epsilon$.

Thus, after integrating by parts we obtain

$$\int_{\partial D_\epsilon} f \wedge B_0 =$$

$$\int_{\partial D_\epsilon} f \wedge \Omega_0(\alpha_\epsilon) + \int_{\partial D_\epsilon} \bar{\partial} f \wedge C^\epsilon + \int_{\partial D_\epsilon} f \wedge \mathcal{E}_{2-2n}$$

$$+ \sum_{\mu=0}^{n-2} \int_{D_\epsilon} \bar{\partial}_\zeta \left[f \wedge \left(\bar{\partial}_\zeta \left(\frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} \right) + r_\epsilon^* \frac{\mathcal{E}_{2+2\mu} \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + \frac{\mathcal{E}_1}{\phi_\epsilon^n} \right) \right]$$

$$+ \sum_{\mu=0}^{n-2} \int_{\partial D_\epsilon} f \wedge \left(R_1 \frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} \rho^{2n}} + R_1 \frac{\mathcal{E}_{5+2\mu}}{\phi_\epsilon^{2+\mu} \rho^{2n}} \right),$$

where we use an application of Stoke's Theorem in the second integral on the right hand side.

We now replace all occurrences of ρ^2 in the denominators by P_ϵ, since the two are equal on ∂D_ϵ, in the remaining boundary integrals, and then change those

boundary integrals to volume integrals by Stoke's Theorem:

$$\int_{\partial D_\epsilon} f \wedge B_0 =$$

$$\int_{D_\epsilon} \bar{\partial} f \wedge \Omega_0(\alpha_\epsilon) + \int_{D_\epsilon} f \wedge \bar{\partial}_\zeta \Omega_0(\alpha_\epsilon) - \int_{D_\epsilon} \bar{\partial} f \wedge \bar{\partial}_\zeta C^\epsilon + \int_{D_\epsilon} \bar{\partial} f \wedge \mathcal{E}_{2-2n}$$

$$+ \int_{D_\epsilon} f \wedge \mathcal{E}_{1-2n} + \sum_{\mu=0}^{n-2} \int_{D_\epsilon} \bar{\partial} f \wedge \left(\frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + R_1 \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{2+\mu} P_\epsilon^n} \right.$$

$$+ \frac{\mathcal{E}_{5+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} + R_1 \frac{\mathcal{E}_{4+2\mu} r_\epsilon^*}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} + R_1 \frac{\mathcal{E}_{2+2\mu} r_\epsilon^*}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + \frac{\mathcal{E}_1}{\phi_\epsilon^n} \right)$$

$$+ \sum_{\mu=0}^{n-2} \int_{D_\epsilon} f \wedge \left(R_1 \frac{\mathcal{E}_{1+2\mu} r_\epsilon^*}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + R_1 \frac{\mathcal{E}_{3+2\mu} r_\epsilon^*}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} + \frac{\mathcal{E}_0}{\phi_\epsilon^n} + \frac{\mathcal{E}_2 + \mathcal{E}_1 \wedge \bar{\partial} r_\epsilon}{\phi_\epsilon^{n+1}} \right)$$

$$+ \sum_{\mu=0}^{n-2} \int_{D_\epsilon} f \wedge \left(\frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + R_1 \frac{\mathcal{E}_{2+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^n} + R_2 \frac{\mathcal{E}_{3+2\mu}}{\phi_\epsilon^{2+\mu} P_\epsilon^n} \right.$$

$$\left. + R_1 \frac{\mathcal{E}_{4+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} + \frac{\mathcal{E}_{5+2\mu}}{\phi_\epsilon^{1+\mu} P_\epsilon^{n+1}} \right).$$

Inserting this expression of the boundary integral into (4.2), and using our notation of operators of a certain type, we can write

$$f(z) = \int_{D_\epsilon} f \wedge \bar{\partial}_\zeta \Omega_0(\alpha_\epsilon) + \int_{D_\epsilon} \bar{\partial} f \wedge \Omega_0(\alpha_\epsilon) - \int_{D_\epsilon} \bar{\partial} f \wedge \bar{\partial}_\zeta C^\epsilon$$

$$- \int_{D_\epsilon} \bar{\partial} f \wedge B_0^\epsilon$$

$$+ (f, \mathcal{E}_{1-2n}) + \left(f, \mathcal{A}_{(0,1)}^\epsilon \right) + (\bar{\partial} f, \mathcal{E}_{2-2n}) + \left(\bar{\partial} f, \mathcal{A}_{(0,2)}^\epsilon \right).$$

Then as in [7], a rearrangement of terms leads to the representation for $f \in C^1(\overline{D})$ and $z \in D_\epsilon$

$$f = \mathbf{T}_0^\epsilon \bar{\partial} f + \mathbf{P}_0^\epsilon f + (A_{(0,2)}^\epsilon + E_{2-2n}) \bar{\partial} f + (A_{(0,1)}^\epsilon + E_{1-2n}) f.$$

We then let $\epsilon \to 0$ and the right hand side approaches the right hand side of (4.1). This establishes (4.1) for $f \in C^1(\overline{D})$.

To see the theorem is valid for $f \in L^2(D) \cap \mathrm{Dom}\, \bar{\partial}$, we refer to Theorems 3.3, 3.4, and 3.5 to show all operators in (4.1) map L^p to L^p for $p \geq 2$. □

5. A SYMMETRY ARGUMENT

From [3] we have the

Proposition 5.1.

$$\mathbf{P}_0 - \mathbf{P}_0^* = A_{(0,1)}.$$

We denote by $\mathcal{S}_\infty(x,y)$ a kernel which has the properties that $\forall \lambda > 0$

$$\int_D |\mathcal{S}_\infty(x,y)|^\lambda dV(x) < \infty$$

$$\int_D |\mathcal{S}_\infty(x,y)|^\lambda dV(y) < \infty,$$

and which is also smoothing in the sense that $\forall k$

$$\|(\gamma T)^k \mathbf{S}_\infty f\|_{L^\infty} \lesssim \|f\|_2,$$

where $\mathbf{S}_\infty$ is an operator associated with a $\mathcal{S}_\infty$ kernel. We have the mapping property

$$\mathbf{S}_\infty : L^2(D) \to L^p(D)$$

$\forall p \geq 2$ by the generalized Young's inequality [8].

Lemma 5.2.

$$\bar{\partial}_x R_1 \mathcal{P}_0^* = \bar{\partial}_x \mathcal{A}_1 + \mathcal{S}_\infty.$$

Proof. $\mathcal{P}_0^*$ has the form

$$\mathcal{P}_0(x,y)^* = R_2(y)\frac{l(y)}{\phi(y,x)^{n+1}} + \mathcal{A}_{(0,1)},$$

where $l(y)$ is smooth and real-valued, and we use $R_1(y) = R_1(x) + \mathcal{E}_1$. Thus

$$\bar{\partial}_x R_1 \mathcal{P}_0^* = \bar{\partial}_x \left(R_3(y)\frac{l(y)}{\phi(y,x)^{n+1}} \right) + \bar{\partial}_x \mathcal{A}_1$$

$$= \bar{\partial}_x \mathcal{A}_1 + \mathcal{S}_\infty,$$

since

$$\bar{\partial}_x \left(R_3(y)\frac{l(y)}{\phi(y,x)^{n+1}} \right) = \mathcal{S}_\infty,$$

due to the fact that $\phi(y,x)$ is holomorphic in the x variable near $x = y$. $\qquad\square$

We let N be the $\bar{\partial}$-Neumann operator on $(0,1)$-forms, and we make use of the relation

$$B = I - \bar{\partial}^* N \bar{\partial}$$

to obtain

Corollary 5.3.

$$\mathbf{P}_0^* R_1(f - Bf) = A_1(f - Bf) + \mathbf{S}_\infty N \bar{\partial} f.$$

Proof.

$$\mathbf{P}_0^* R_1(f - Bf) = (f - Bf, R_1 \mathcal{P}_0^*)$$

$$= (\bar{\partial}^* N \bar{\partial} f, R_1 \mathcal{P}_0^*)$$

$$= (N \bar{\partial} f, \bar{\partial}_\zeta R_1 \mathcal{P}_0^*)$$

$$= (N \bar{\partial} f, \bar{\partial}_\zeta \mathcal{A}_1 + \mathcal{S}_\infty)$$

$$= (\bar{\partial}^* N \bar{\partial} f, \mathcal{A}_1) + \mathbf{S}_\infty N \bar{\partial} f.$$

$$\square$$

6. Estimates

We apply Theorem 4.1 to $f - Bf$.

$$f - Bf = \mathbf{T}_0 \bar{\partial} f + \mathbf{P}_0(f - Bf) + (A_{(0,2)} + E_{2-2n})\bar{\partial} f + (A_{(0,1)} + E_{1-2n})(f - Bf),$$

and then we use Proposition 5.1 to obtain

$$f - Bf = \mathbf{T}_0 \bar{\partial} f + \mathbf{P}_0^*(f - Bf) + (A_{(0,2)} + E_{2-2n})\bar{\partial} f + (A_{(0,1)} + E_{1-2n})(f - Bf).$$

Then after applying Corollary 5.3, we write

Theorem 6.1.

(6.1) $\quad f - Bf = (\mathbf{T}_0 + A_{(0,2)} + E_{2-2n})\bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + (A_{(0,1)} + E_{1-2n})(f - Bf).$

We introduce the notion of Z operators. We denote by Z_1 those operators which are of a type ≥ 1 and $\epsilon = 0$ (that is, we consider operators on spaces of the domain, D) or are $E_{1-2n} \circ R_1$ and also the sum of such operators. We further include in our notation of Z_1 those operators whose kernels conform to our definition of admissible with the factors, R_1 being replaced by a factor of γ. Thus, for instance, a kernel of smooth type 1 multiplied by γ^2 will be a Z_1 operator. $E_{1-2n} \circ \gamma$ will also be a Z_1 operator.

Let $\mathcal{Z}_1(x, y)$ be a kernel of a Z_1 operator. We define Z_j operators to be those operators which have kernels of the form $\mathcal{Z}_1(x, y)\mathcal{E}_{j-1}(x, y)$. We establish mapping properties for Z_j operators.

For the proof of the properties of Z_j operators, we will refer to a lemma of Schmalz (see the proof of Lemma 3.2 in [9]) which provides a useful coordinate system in which to prove estimates.

Lemma 6.2. *Let* $E_\delta(y) := \{x \in D : \rho(x, y) < \delta \gamma(y)\}$ *for* $\delta > 0$*. Then there is a constant* c *and a number,* $m \in \{1, \dots, 2n\}$ *such that for all* $y \in D$*, there are local coordinates around each* $x \in E_c(y)$ *given by* $\zeta_j = t_j + it_{j+n}$ *for* $1 \leq j \leq n$*, and*

$$\{-r(\zeta), t_1, \dots, \hat{}_m \dots, t_{2n}\},$$

where t_m *is omitted, forms a coordinate system near* x*. We have the estimate*

(6.2) $$dV \lesssim \frac{1}{\gamma(z)} |dr(\zeta) \wedge dt_1 \wedge \dots \hat{}_m \dots \wedge dt_{2n}|,$$

where dV *is the Euclidean volume form on* $\mathbb{R}^{2n}$*.*

Lemma 6.3. *For* $2 \leq p < \infty$ *and* $1 \leq j \leq n + 1$

(6.3) $$\|Z_j f\|_{L^p} \lesssim \|f\|_{L^{p,jp}}.$$

Let $0 < \epsilon' < \epsilon$*.*

(6.4) $$\|Z_j f\|_{L^{\infty, \epsilon, 0}} \lesssim \|f\|_{L^{\infty, j+\epsilon', 0}}.$$

Proof. We prove (6.3) for kernels of the form $\mathcal{A}_{(1,1)}(x, y)\mathcal{E}_{j-1}$. We show below that $\mathcal{A}_{(1,1)}(x, y)$ satisfies

(6.5) $$\sup_{y \in \Omega} \int \frac{1}{\gamma^j(x)} |\mathcal{A}_{(1,1)}(x, y)\mathcal{E}_{j-1}(x, y)||r(x)|^{-\delta}|r(y)|^\delta dV(x) < \infty$$

for $\delta < 1/2$. The lemma then follows from the generalized Young's inequality.

We further restrict our proof to the cases in which $\mathcal{A}_{(1,1)}$ satisfies

$$i) \ \frac{1}{\gamma(x)}|\mathcal{A}_{(1,1)}| \lesssim \gamma(x)\frac{1}{P^{n-1/2-\mu}|\phi|^{\mu+1}} \qquad \mu \geq 1$$

$$ii) \frac{1}{\gamma(x)}|\mathcal{A}_{(1,1)}| \lesssim \frac{1}{P^{n-1-\mu}|\phi|^{\mu+1}} \qquad \mu \geq 1$$

$$iii) \frac{1}{\gamma(x)}|\mathcal{A}_{(1,1)}| \lesssim \frac{1}{\gamma(x)}\frac{1}{P^{n-3/2-\mu}|\phi|^{\mu+1}} \qquad \mu \geq 1.$$

We will prove the more difficult case $iii)$, as cases $i)$ and $ii)$ follow similar arguments, and we leave the details of those cases to the reader.

We denote the critical points of r by $p_1, \ldots, p_k$, and take ε small enough so that in each

$$U_{2\varepsilon}(p_j) = \{x : D \cap \rho(x, p_j) < 2\varepsilon\},$$

for $j = 1, \ldots, k$, there are coordinates $u_{j_1}, \ldots, u_{j_m}, v_{j_{m+1}}, \ldots, v_{j_{2n}}$ such that

$$(6.6) \qquad\qquad -r(x) = u_{j_1}^2 + \cdots + u_{j_m}^2 - v_{j_{m+1}}^2 - \cdots - v_{j_{2n}}^2,$$

with $u_{j_\alpha}(p_j) = v_{j_\beta}(p_j) = 0$ for all $1 \leq \alpha \leq m$ and $m+1 \leq \beta \leq 2n$, from the Morse Lemma.

We divide the estimates into subcases depending on whether $y \in U_\varepsilon$.

Subcase $a)$. Suppose $y \in U_\varepsilon(p_j)$. We estimate

$$(6.7) \qquad\qquad \int_{U_{2\varepsilon}(p_j)} \frac{\rho^{j-1}(x,y)}{\gamma^j(x)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(x)|^\delta}dV(x)$$

and

$$(6.8) \qquad\qquad \int_{D_\epsilon \setminus U_{2\varepsilon}} \frac{\rho^{j-1}(x,y)}{\gamma^j(x)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(x)|^\delta}dV(x).$$

We break up the integral in (6.7) into integrals over $E_c(y)$ and its complement, where c is as in Lemma 6.2, and we choose c small enough so that, in $E_c(y)$, we have $\rho(x,y) \lesssim \gamma(x)$.

In the case $U_{2\varepsilon}(p_j) \cap E_c(y)$, we use a coordinate system, $s = -r(\zeta)$, $t_1, \ldots, t_{2n-1}$, and estimate

$$(6.9) \quad \int_{U_{2\varepsilon}(p_j) \cap E_c(y)} \frac{1}{\gamma(x)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(x)|^\delta}dV(x)$$

$$\lesssim \int_{\mathbb{R}_+^2} \frac{t^{2n-2}}{\gamma(\zeta)\gamma(z)s^\delta(\theta+s+t^2)^{\mu+1}(s+t)^{2n-3-2\mu}}dsdt$$

$$\lesssim \int_{\mathbb{R}_+^2} \frac{t^{2\mu-1}}{s^\delta(\theta+s+t^2)^{\mu+1}}dsdt$$

$$\lesssim \int_{\mathbb{R}_+^2} \frac{1}{s^\delta(\theta^{1/2}+s^{1/2}+t)^3}dsdt$$

$$\lesssim \int_0^\infty \frac{1}{s^\delta(\theta+s)}dsdt$$

$$\lesssim \frac{1}{\theta^\delta},$$

where we use the notation $\mathbb{R}_+^j = \overbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}^{j \text{ times}}$.

We now estimate the integral

$$(6.10) \qquad \int_{U_{2\varepsilon}(p_j)\backslash E_c(y)} \frac{\rho^{j-1}(x,y)}{\gamma^j(x)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(x)|^\delta}dV(x).$$

We can assume, without loss of generality that in $U_{2\varepsilon}(p_j)$ there are coordinate charts ζ and z for x and y respectively. We define $w_1,\ldots,w_{2n}$ by

$$w_\alpha = \begin{cases} u_{j_\alpha} & \text{for } 1 \le \alpha \le m \\ v_{j_\alpha} & \text{for } m+1 \le \alpha \le 2n, \end{cases}$$

and we let $t_1,\ldots,t_{2n}$ be defined by $\zeta_\alpha = t_\alpha + it_{n+\alpha}$. From the Morse Lemma, the Jacobian of the transformation from coordinates $t_1,\ldots,t_{2n}$ to $w_1,\ldots,w_{2n}$ is bounded from below and above and thus we have

$$|\zeta - z| \simeq |w(\zeta) - w(z)|$$

for $\zeta, z \in U_{2\varepsilon}(p_j)$.

From (6.6) we have $\gamma(z) \gtrsim |w(z)|$, and thus

$$\begin{aligned} |w(\zeta) - w(z)| &\simeq |\zeta - z| \\ &\gtrsim \gamma(z) \\ &\gtrsim |w(z)| \\ &\ge |w(\zeta)| - |w(\zeta) - w(z)|, \end{aligned}$$

and we obtain

$$\begin{aligned} |w(\zeta)| &\lesssim |w(\zeta) - w(z)| \\ &\simeq |\zeta - z|. \end{aligned}$$

Thus, $|w(\zeta)| \lesssim |\zeta - z|$ and $|w(\zeta)| \lesssim \gamma(\zeta)$ for $\zeta, z \in U_{2\varepsilon}(p_j)$.

We can therefore bound the integral in (6.10) by

$$\int_{U_{2\varepsilon}(p_j)\backslash E_c(z)} \frac{|\zeta - z|^{j-1}}{\gamma^j(\zeta)|\phi|^{\mu+1}P^{n-3/2-\mu}|r(\zeta)|^\delta}dV(\zeta)$$

$$\lesssim \int_{U_{2\varepsilon}(p_j)\backslash E_c(z)} \frac{1}{\gamma^j(\zeta)|\phi|^{1/2}|\zeta - z|^{2n-1-j}|r(\zeta)|^\delta}dV(\zeta)$$

$$\lesssim \int_V \frac{u^{m-1}v^{2n-m-1}}{(u+v)^{2n-1}(\theta + u^2 + v^2)^{1/2}(u^2 - v^2)^\delta}dudv$$

$$(6.11) \qquad \lesssim \int_V \frac{1}{u(\theta + u^2)^{1/2}(u^2 - v^2)^\delta}dudv,$$

where V is a bounded region. We make the substitution $v = \tilde{v}u$, since $v^2 < u^2$, and write (6.11) as

$$\int_0^M \frac{1}{u^{2\delta}(\theta + u^2)^{1/2}}du \int_0^1 \frac{1}{(1 - \tilde{v}^2)^\delta}d\tilde{v} \lesssim \frac{1}{\theta^\delta} \int_0^M \frac{1}{u^{2\delta}(1 + u^2)^{1/2}}du$$

$$\lesssim \frac{1}{\theta^\delta},$$

where $M > 0$ is some constant. We have therefore bounded (6.7), and we turn now to (6.8).

In $D \setminus U_{2\varepsilon}$ we have that $\rho(x,y)$ and $\gamma(x)$ are bounded from below so

$$\int_{D \setminus U_{2\varepsilon}} \frac{\rho^{j-1}(x,y)}{\gamma^j(x)|\phi|^{\mu+1} P^{n-3/2-\mu}|r(x)|^\delta} dV(x) \lesssim \int_{D \setminus U_{2\varepsilon}} \frac{1}{|r(x)|^\delta} dV(x) \lesssim 1,$$

the last inequality following because in $D \setminus U_{2\varepsilon}$ r can be chosen as a coordinate since $\gamma(x)$ is bounded from below. This finishes subcase a).

Case b). Suppose $y \notin U_\varepsilon$. We divide D into the regions $D \cap E_c(y)$ and $D \setminus E_c(y)$.

In $D \cap E_c(y)$ the same coordinates and estimates work here as in establishing the estimates for the integral in (6.9).

In $D \setminus E_c(y)$ we have $\rho(x,y) \gtrsim \gamma(y)$, but $\gamma(y)$ is bounded from below, since $y \notin U_\varepsilon$. We therefore have to estimate

$$\int_D \frac{1}{\gamma^j(x)|r(x)|^\delta} dV(x),$$

which is easily done by working locally with the coordinates $w_1, \ldots, w_{2n}$ above.

(6.4) is proved similarly. $\qquad\square$

Applying R_1 to (6.1) gives us

$$(6.12) \qquad R_1(y)(f - Bf) = Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + Z_1(f - Bf).$$

An iteration leads to

Proposition 6.4.

$$(6.13) \quad R_k(y)(f - Bf) = Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + \sum_{j=1}^{k} \sum_{i_1 + \cdots + i_j = k} Z_{i_1} \circ \cdots \circ Z_{i_j}(f - Bf).$$

Proof. The case $k = 1$ is just Equation 6.12. Then, given (6.13) for a particular k, we prove the corresponding equation $R_{k+1}(y)(f - Bf)$ by multiplying (6.13) by a factor of R_1:

$$R_{k+1}(y)(f - Bf) = Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + \sum_{j=1}^{k} \sum_{i_1 + \cdots + i_j = k+1} Z_{i_1} \circ \cdots \circ Z_{i_j}(f - Bf)$$

$$+ \sum_{j=1}^{k} \sum_{i_1 + \cdots + i_j = k} Z_{i_1} \circ \cdots \circ Z_{i_j} R_1(x)(f - Bf)$$

$$= Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + \sum_{j=1}^{k} \sum_{i_1 + \cdots + i_j = k+1} Z_{i_1} \circ \cdots \circ Z_{i_j}(f - Bf)$$

$$+ \sum_{j=1}^{k} \sum_{i_1 + \cdots + i_j = k} Z_{i_1} \circ \cdots \circ Z_{i_j} \circ Z_1(f - Bf)$$

$$= Z_1 \bar{\partial} f + \mathbf{S}_\infty N \bar{\partial} f + \sum_{j=1}^{k+1} \sum_{i_1 + \cdots + i_j = k+1} Z_{i_1} \circ \cdots \circ Z_{i_j}(f - Bf),$$

where we use

$$R_1(y) Z_i = Z_i \circ R_1(x) + Z_{i+1}$$

in the first step. We also use the commutator relations below (see Theorem 6.5) to establish that $Z_j \circ \mathbf{S}_\infty = \mathbf{S}_\infty$. $\qquad\square$

When dealing with derivatives in order to establish the C^k-estimates we will need to know how certain vector fields commute with our Z_j operators. We make the important remark here that in the coordinate patch of a critical point, the smooth tangential vector fields are not smooth combinations of derivatives with respect to the coordinate system described in Lemma 6.3 (see 6.6). To deal with this difficulty, we define $\mathcal{E}_{j,k}(x,y)$, for $j \geq 0$, for those double forms on open sets $U \subset X \times X$ such that $\mathcal{E}_{j,k}$ is smooth on U and satisfies

$$\mathcal{E}_j(x,y) \lesssim \xi(x)\rho^j(x,y),$$

where $\xi(x)$ is a smooth function on U satisfying

$$|\gamma^k D_k \xi| \lesssim 1,$$

for D_k a kth order differential operator.

In a neighborhood of a critical point, the manner in which we defined smooth tangential vector fields leads to combinations of derivatives with respect to the coordinates of (6.6) with coefficients only in $C^0(\overline{D})$ due to factors of γ which occur in the denominators of such coefficients. In general a kth order derivative of such coefficients is in $\mathcal{E}_{0,-k}$. Thus, when integrating by parts, special attention has to be paid to these non-smooth terms. We obtain the following theorem from [2].

Theorem 6.5. *Let X be a smooth tangential vector field. Then*

$$\gamma^* X^y Z_1 = -Z_1 \tilde{X}^x \gamma + Z_1^{(0)} + \sum_{\nu=1}^{l} Z_1^{(\nu)} W_\nu^x \gamma,$$

where $\tilde{X}$ is the adjoint of X, the $Z_1^{(\nu)}$ are also Z_1 operators, and $W^x \in T_x^{1,0}(\partial D) \oplus T_x^{0,1}(\partial D)$.

We need the

Proposition 6.6. *Let $p \geq 2$, $s > p$ and $k(s)$ an integer which satisfies*

$$\frac{1}{s} > \frac{1}{p} - \frac{k(s)}{2n+2}.$$

There are $0 < \epsilon, \epsilon'$ small enough such that

i) $\displaystyle\sum_{j=1}^{k} \sum_{i_1+\cdots+i_j=k} Z_{i_1} \circ \cdots \circ Z_{i_j} : L^p(D) \to L^s(D)$

ii) $\displaystyle\gamma T \sum_{j=1}^{n+2} \sum_{i_1+\cdots+i_j=n+2} Z_{i_1} \circ \cdots \circ Z_{i_j} : L^{\infty,n+2+\epsilon,0}(D) \to L^{\infty,\epsilon',0}(D) \qquad \epsilon' < \epsilon$

iii) $Z_j : L^{\infty,m+j+\epsilon,0}(D) \to L^{\infty,m+\epsilon',0}(D) \qquad \epsilon' < \epsilon.$

Proof. *i)* We use Theorem 3.3 *ii)* and Theorem 3.5 *ii)* to conclude

$$Z_{i_1} \circ \cdots \circ Z_{i_j} : L^p(D) \to L^s(D),$$

where

$$\frac{1}{s} > \frac{1}{p} - \frac{i_1+\cdots+i_j}{2n+2}.$$

ii). We first consider the case of $i_1 = \ldots = i_{n+2} = 1$. We use parts of Theorems 3.3 and 3.5 and part *iii)* of the current theorem to show the composition of operators maps $L^{\infty,n+2+\epsilon,0}$ into $L^{\infty,\epsilon',0}$. We can see this by using the commutator relations

and considering the two compositions $Z_1 \circ \gamma T A_1 \circ Z_1$, and $Z_1 \circ \gamma T E \circ Z_1$. From Theorems 3.3 and 3.5 we can find $\epsilon_1, \ldots, \epsilon_4$ such that $\epsilon_{j+1} < \epsilon_j$ and in the first case we have

$$\|Z_1 \circ \gamma T A_1 \circ Z_1 f\|_{L^{\infty,\epsilon_1,0}} \lesssim \|\gamma T A_1 \circ Z_1 f\|_{L^{\infty,\epsilon_2,\delta}} \lesssim \|Z_1 f\|_{L^{\infty,2+\epsilon_3,0}} \lesssim \|f\|_{L^{\infty,3+\epsilon_4,0}},$$

and, in the second,

$$\|Z_1 \circ \gamma T E \circ Z_1 f\|_{L^{\infty,\epsilon_1}} \lesssim \|\gamma T E \circ Z_1 f\|_{L^{\infty,1+\epsilon_2,0}} \lesssim \|Z_1 f\|_{\Lambda_{\alpha,3-\epsilon_3}} \lesssim \|f\|_{L^{\infty,3+\epsilon_4,0}},$$

where the second and third inequalities are proved in the same way as Theorem 3.5 ii) and iii).

We now consider the case in which $\max\{i_1, \ldots, i_j\} \geq 2$, and we show

$$\gamma^* T Z_j : L^{\infty,j+\epsilon,0} \to L^{\infty,\epsilon'}$$

for $\epsilon < \epsilon'$

For $T = T^y$ a smooth first order tangential differential operator on D, with respect to the y variable, we have

$$T\gamma(y) \lesssim 1$$
$$Tr = 0$$
$$Tr^* = \mathcal{E}_0 r$$
$$TP = \mathcal{E}_1 + \mathcal{E}_0 r r^*$$
$$T\phi = R_1 \mathcal{E}_0 + \mathcal{E}_1.$$

We can therefore write for $j \geq 2$

(6.14) $$\gamma^* T Z_j = Z_{j-2}\gamma^2 + Z_{j-1}\gamma + Z_j,$$

where we use the convention that Z_0 refers to an operator of type ≥ 0.

Noting that, as a result of the commutator relations, and our definition of Z_j we also have for $j \geq 2$

(6.15) $$\gamma^* T Z_j = Z_{j-1} \circ \gamma + Z_j + Z_j \circ \gamma T.$$

We again use the commutator relations, (6.15), and Lemma 6.3 to reduce the proof to the examination of

$$Z_{i_1} \circ \cdots \circ \gamma T Z_{i_k} \circ \cdots \circ Z_{i_j}$$

where $i_k \geq 2$. But from (6.14) we can write

$$Z_{i_1} \circ \cdots \circ (Z_{i_k-2} \circ \gamma^2 + Z_{i_k-1} \circ \gamma + Z_{i_k}) \circ \cdots \circ Z_{i_j},$$

and since $i_1 + \cdots + i_{k-1} + i_{k+1} + \cdots + i_j = n - k + 2$, this case follows from Theorem 3.4, Lemma 6.3 and property iii) of the current theorem.

iii). This is just a corollary of Lemma 6.3. $\qquad\square$

We first establish the L^p-estimates. The next theorem was established in [3] in the case of strictly pseudoconvex domains in $\mathbb{C}^n$, (see Theorem 5.2 of [3]).

Theorem 6.7. *For $p \geq 2$ let $k = k(p)$ satisfy*

$$\frac{1}{p} > \frac{1}{2} - \frac{k}{2n+2}.$$

Then

$$\|Bf\|_{L^{p,kp}} \lesssim \|f\|_{L^p}.$$

Proof. We start by multiplying (4.1) by R_k and subtracting (6.13):

$$(6.16) \quad R_k B f = A_0 f + \mathbf{S}_\infty N \bar\partial f$$

$$+ \sum_{j=1}^{k} \sum_{i_1 + \cdots + i_j = k} Z_{i_1} \circ \cdots \circ Z_{i_j} f + \sum_{j=1}^{k} \sum_{i_1 + \cdots + i_j = k} Z_{i_1} \circ \cdots \circ Z_{i_j} B f.$$

The L^p norm of the first term on the right hand side is bounded by $\|f\|_p$ by Theorem 3.4 *ii*).

We use the fact that

$$\mathbf{S}_\infty : L^2 \to L^p$$

for any $p \geq 2$ and that

$$\|N\bar\partial f\|_{L^2} \lesssim \|f\|_{L^2} \lesssim \|f\|_{L^p}$$

(see [5]) to handle the second term.

For the last two sums in (6.16) we use Proposition 6.6 *i*) to show

$$Z_{i_1} \circ \cdots \circ Z_{i_j} : L^2 \to L^p,$$

where p is given by

$$(6.17) \qquad\qquad \frac{1}{p} > \frac{1}{2} - \frac{k}{2n+2}.$$

Hence, the L^p norm of the last two terms in (6.16) is bounded by

$$\|f\|_{L^2} + \|Bf\|_{L^2} \lesssim \|f\|_{L^2} \lesssim \|f\|_{L^p}$$

which we use to finish the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that the right hand side of (6.17) is 0 for $k(p) = n + 1$. Thus, we have the

Corollary 6.8. *For $2 \leq p < \infty$ we have the weighted estimates*

$$\|Bf\|_{L^p, (n+1)p} \lesssim \|f\|_{L^p}.$$

For the C^k-estimates we take $k = n + 2$ in (6.13) and use Proposition 6.6 *i*) above to establish

$$(6.18) \qquad\qquad \|\gamma^{n+2}(f - Bf)\|_{L^\infty} \lesssim \|\bar\partial f\|_{L^\infty} + \|f\|_2.$$

We use the notation D^k to denote a k-th order differential operator, which is a sum of terms which are composites of k vector fields. We define

$$Q_k(f) = \sum_{j=0}^{k} \|\gamma^j D^j \bar\partial f\|_\infty + \|f\|_2.$$

T^k will be used for a k-th order tangential differential operator, which is a sum of terms which are composites of k tangential vector fields.

Lemma 6.9. *Let T^k be a tangential operator of order k. Let $\epsilon > 0$, then*

$$\|\gamma^{(n+2)+\epsilon+k} T^k (f - Bf)\|_{L^\infty} \lesssim Q_k(f).$$

Proof. The proof is by induction. The first step, $k = 0$ is contained in (6.18).

We start with (6.13) with $k = n + 2$:

$$R_{n+2}(y)(f - Bf) = Z_1 \bar\partial f + \mathbf{S}_\infty N \bar\partial f + \sum_{j=1}^{n+1} \sum_{i_1 + \cdots + i_j = n+2} Z_{r_{i_1}} \circ \cdots \circ Z_{r_{i_j}} (f - Bf)$$

and apply $\gamma^\epsilon \gamma^k T^k$ to both sides. After employing the commutator relations in Theorem 6.5 and the induction step, we have

(6.19)

$$\gamma^\epsilon R_{n+2} \gamma^k T^k (f - Bf) = \gamma^\epsilon \sum_{j=0}^{k} (Z_1 \gamma^j T^j)(\bar{\partial} f) + \gamma^\epsilon \mathcal{S}_\infty N \bar{\partial} f$$

$$+ \gamma^\epsilon \sum_{m=1}^{k-1} \sum_{j=1}^{n+2} \sum_{i_1 + \cdots + i_j = n+2} Z_{r_{i_1}} \circ \cdots \circ Z_{r_{i_j}} \gamma^m T^m (f - Bf)$$

$$+ \gamma^\epsilon \gamma T \sum_{j=1}^{n+2} \sum_{i_1 + \cdots + i_j = n+2} Z_{r_{i_1}} \circ \cdots \circ Z_{r_{i_j}} \gamma^{k-1} T^{k-1} (f - Bf).$$

The L^∞ norms of the first three terms on the right of (6.19) are bounded by $Q_k(f)$ by Proposition 6.6 i) and the induction step. For the last term we use Proposition 6.6 ii) to show its L^∞ norm is bounded by

$$\|\gamma^{(n+2)+\epsilon'+k-1} T^{k-1} (f - Bf)\|_\infty \lesssim Q_{k-1}(f),$$

for $\epsilon' < \epsilon$, by the induction hypothesis. $\square$

In order to generalize Lemma 6.9 to include non-tangential operators, we express a normal derivative of a component of a function, f, in terms of tangential operators acting on f and components of $\bar{\partial} f$,

We have the decomposition in the following form:

(6.20) $$\gamma N f = \sum_j a_j \gamma T_j f + \sum_j b_j f + \sum_j c_j \gamma (\bar{\partial} f)_j,$$

where the coefficients a_j, b_j, and c_j are all of the form $\mathcal{E}_{0,0}$. The decomposition is well known in the smooth case (see [6]) and to verify (6.20) in a neighborhood of $\gamma = 0$, one may use the coordinates $u_{j_1}, \ldots, u_{j_m}, v_{j_{m+1}}, \ldots, v_{j_{2n}}$ as in (6.6) above.

It is then straightforward how to generalize Lemma 6.9. Suppose D^k is a kth order differential operator which contains the normal field at least once. In $\gamma^k D^k$ we commute γN with terms of the form γT, where T is tangential, and we consider the operator $D^k = D^{k-1} \circ \gamma N$, where D^{k-1} is of order $k - 1$. The error terms due to the commutation involve differential operators of order $\leq k - 1$. From (6.20) we just have to consider $D^{k-1} \gamma T f$ and $D^{k-1} \bar{\partial} f$. The last two terms are bounded by $Q_{k-1}(f)$, and we repeat the process with $D^{k-1} \gamma T f$, until we are left with k tangential operators for which we can apply Lemma 6.9. We thereby obtain the weighted C^k estimates given in Theorem 1.2.

Theorem 6.10. *For $\epsilon > 0$*

$$\|\gamma^{(n+2)+\epsilon+k}(f - Bf)\|_{C^k} \lesssim Q_k(f).$$

As an immediate consequence we obtain weighted C^k estimates for the canonical solution, the solution of minimal L^2 norm, to the $\bar{\partial}$-equation. Let v be any solution to $\bar{\partial} v = f$ with L^2 estimates, the existence being guaranteed by Hörmander's solution [5]. Then $u = v - Bv$ is the canonical solution to $\bar{\partial} u = f$, and from above we obtain the following estimates.

Corollary 6.11. *The canonical solution to $\bar{\partial} u = f$ satisfies*

$$\|\gamma^{(n+2)+\epsilon+k} u\|_{C^k} \lesssim \|\gamma^k f\|_{C^k} + \|f\|_2.$$

References

1. D. Ehsani. Integral representations on non-smooth domains. preprint.
2. D. Ehsani. Weighted C^k estimates for a class of integral operators on non-smooth domains. preprint.
3. D. Ehsani and I. Lieb. L^p-estimates for the Bergman projection on strictly pseudoconvex non-smooth domains. *Math. Nach.*, 281:913–1062, 2008.
4. G. Henkin and J. Leiterer. *Theory of Functions on Complex Manifolds*. Monographs in Math. Berkhäuser, Basel.
5. L. Hörmander. L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta. Math.*, 113:89–152, 1965.
6. I. Lieb and J. Michel. *The Cauchy-Riemann complex*, volume E 34 of *Aspects of Mathematics*. Vieweg, Wiesbaden, 2002.
7. I. Lieb and R. Range. Integral representations and estimates in the theory of the $\bar{\partial}$-Neumann problem. *Ann. Math.*, 123:265–301, 1986.
8. R. M. Range. *Holomorphic functions and integral representations in several complex variables*, volume 108 of *Graduate Texts in Mathematics*. Springer, Berlin, 1986.
9. G. Schmalz. Solution of the $\bar{\partial}$-equation with uniform estimates on strictly q-convex domains with non-smooth boundary. *Math. Z.*, 202:409–430, 1989.

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